1. (a) POINTS IN THE PLANE REPRESENT POSITIONS; → HAVE POSITIONS, BUT NO MAGNITUDE OR DIRECTION

VECTORS IN THE PLANE REPRESENT DISPLACEMENTS. → HAVE MAGNITUDE & DIRECTION, BUT NO POSITION

*THESE CONCEPTS ARE IN NO WAY SIMILAR TO ONE ANOTHER IN MEANING!

(b) WE JUST USE LETTERS FOR POINTS (p, q, ETC.),
BUT WE ADORN VECTORS WITH AN ARROW ($\vec{v}, \vec{w}$, ETC.)

(c) WE CONSIDER POINTS TO HAVE "WEIGHT" 1 AND VECTORS TO HAVE "WEIGHT" 0.

(THIS HELPS US TO DISTINGUISH WHAT RESULTS WHEN WE COMBINE POINTS AND/OR VECTORS)

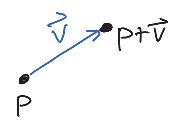
(d) THE FUNDAMENTAL OPERATION ON VECTORS IS THE LINEAR COMBINATION, SCALING EACH VECTOR BY SOME REAL NUMBER AND THEN ADDING THEM UP.

(E.G., $2\vec{v} + 3\vec{w} - \vec{r}$)

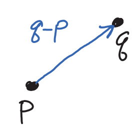
FOR POINTS, THE SUM OF THE COEFFICIENTS MUST BE 1, GIVING WHAT IS CALLED AN AFFINE COMBINATION.

(E.G., $\frac{5}{2}p - \frac{1}{2}q - r$)

(e) ADDING A POINT TO A VECTOR DISPLACES THE POINT:

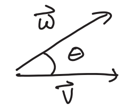


SIMILARLY, SUBTRACTING TWO POINTS GIVES THE DISPLACEMENT "FROM" THE NEGATIVE ONE "TO" THE POSITIVE ONE:



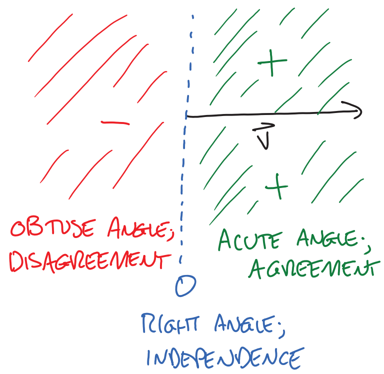
2. IF $\vec{v}, \vec{w}$ ARE VECTORS IN THE PLANE:

(a) $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$, WHERE θ IS THE ANGLE BETWEEN $\vec{v}$ & $\vec{w}$:



$\|\vec{v}\|$ DENOTES THE MAGNITUDE, OR LENGTH, OF $\vec{v}$

THIS CAN BE USED TO:
- COMPUTE LENGTHS: $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$, SO $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$
- COMPUTE ANGLES: $\cos\theta = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$ (AS LONG AS $\|\vec{v}\|, \|\vec{w}\| \neq 0$)
- DETERMINE AGREEMENT: HOW DOES $\vec{w}$ PUSH, RELATIVE TO $\vec{v}$?

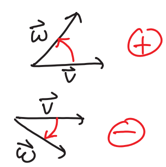


(b) $(\vec{v} \times \vec{w}) \cdot \hat{k} = \pm \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin\theta$, WHERE θ IS THE ANGLE BETWEEN $\vec{v}$ & $\vec{w}$

AND THE $\pm$ SIGN IS TAKEN BY CONSIDERING THE SMALLEST TURN FROM $\vec{v}$ TO $\vec{w}$:

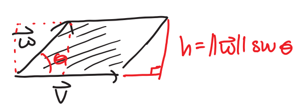
+ IF COUNTERCLOCKWISE,

OR − IF CLOCKWISE

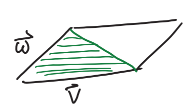


(* NOTE THAT WE COULD ALSO SIMPLY SAY THAT $(\vec{v} \times \vec{w}) \cdot \hat{k} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin\theta$, WHERE θ IS THE DIRECTED ANGLE FROM $\vec{v}$ TO $\vec{w}$, WHERE ↺ IS + AND ↻ IS −. THIS IS FORMALLY THE BEST WAY TO SAY IT, BUT IT DOESN'T STRESS ENOUGH THE IMPORTANCE OF WHEN THE SCALAR CROSS-PRODUCT IS POSITIVE OR NEGATIVE!)

THIS VALUE CAN BE USED TO:
- FIND THE AREA OF THE PARALLELOGRAM FORMED BY $\vec{v}$ & $\vec{w}$:



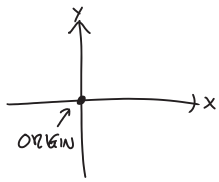
(OR THE AREA OF THE TRIANGLE FORMED BY $\vec{v}$ & $\vec{w}$, WHICH IS HALF OF THIS:



)

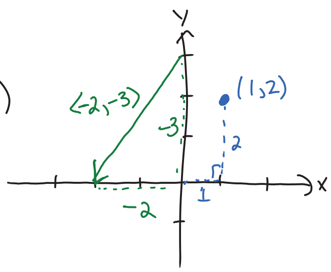
- DETERMINE WHETHER THE TURN FROM $\vec{v}$ TO $\vec{w}$ IS COUNTERCLOCKWISE (+) OR CLOCKWISE (−)

3. (a) TO FORM THE CARTESIAN PLANE, WE CHOOSE A POINT IN THE PLANE AS THE ORIGIN, ALONG WITH X- AND Y-AXES PASSING THROUGH THE ORIGIN AT RIGHT ANGLES TO ONE ANOTHER:



THIS BREAKS THE SYMMETRY OF THE IDEAL EUCLIDEAN PLANE, BUT IT DOES ALLOW US TO NAME POINTS & VECTORS USING PAIRS OF REAL NUMBERS (BELOW) AND TO COMPUTE LINEAR & AFFINE COMBINATIONS, AS WELL AS THE DOT- AND SCALAR CROSS-PRODUCT OF VECTORS.

(b)



POINTS CAN BE NAMED BY THEIR POSITIONS ALONG THE X- & Y-AXES — WE USE (,) TO GROUP THEM.

VECTORS CAN BE NAMED BY THEIR DISPLACEMENTS ALONG THE X- & Y-AXES — WE USE < , > TO GROUP THEM.

* AS POINTS & VECTORS ARE COMPLETELY DIFFERENT OBJECTS (SEE #1!), IT IS CRUCIAL TO KEEP THEM SEPARATED CONCEPTUALLY & NOTATIONALLY, PARTICULARLY AS BOTH ARE REPRESENTED BY A PAIR OF REAL NUMBERS!!

(c) TO SCALE A POINT OR VECTOR, WE SIMPLY SCALE ITS COORDINATES:

$$3\langle 2,-1\rangle = \langle 3\cdot 2,\ 3\cdot -1\rangle = \langle 6,-3\rangle$$

SIMILARLY, TO ADD POINTS AND/OR VECTORS, WE SIMPLY ADD CORRESPONDING COORDINATES:

$$(4,2) + \langle -1,1\rangle = (4+(-1),\ 2+1) = (3,3)$$

COMBINING THESE GIVES US LINEAR & AFFINE COMBINATIONS:

$$\tfrac{1}{3}(4,-2) + \tfrac{2}{3}(2,1) = \left(\tfrac{1}{3}\cdot 4 + \tfrac{2}{3}\cdot 2,\ \tfrac{1}{3}\cdot(-2) + \tfrac{2}{3}\cdot 1\right) = \left(\tfrac{8}{3}, 0\right)$$

(d) THE DOT PRODUCT: $\langle a,b\rangle \cdot \langle x,y\rangle = ax + by$

THE SCALAR CROSS-PRODUCT: $(\langle a,b\rangle \times \langle x,y\rangle)\cdot \hat{k} = ay - bx$

4. AFFINE COORDINATES ADD ONE SPECIAL COORDINATE TO POINTS & VECTORS, GIVING THEIR "WEIGHT":

$$(x,y) \rightsquigarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \langle x,y\rangle \rightsquigarrow \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

POINT VECTOR

* WE CAN FREELY FORM LINEAR COMBINATIONS OF POINTS & VECTORS IN AFFINE COORDINATES, AND THE BOTTOM NUMBER TELLS US WHAT RESULTS: $1 \Leftrightarrow$ POINT, $0 \Leftrightarrow$ VECTOR, AND ANYTHING ELSE $\Leftrightarrow$ NEITHER!

5. (a) $p+q+r$: TOTAL WEIGHT $1+1+1=3$ — NEITHER.

(b) $\frac{1}{3}p+\frac{1}{3}q+\frac{1}{3}r$: TOTAL WEIGHT $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$ — POINT.

(c) $p+\vec{v}$: TOTAL WEIGHT $1+0=1$ — POINT.

(d) $q+3\vec{v}-\vec{w}$: TOTAL WEIGHT $1+0+0=1$ — POINT.

(e) $\vec{r}-5\vec{v}+4\vec{w}$: TOTAL WEIGHT $0+0+0=0$ — VECTOR.

(f) $p+q-r$: TOTAL WEIGHT $1+1-1=1$ — POINT.

(g) $p-\frac{1}{2}q$: TOTAL WEIGHT $1-\frac{1}{2}=\frac{1}{2}$ — NEITHER.

6. (a) $\|\langle 2,-1\rangle\| = \sqrt{\langle 2,-1\rangle\cdot\langle 2,-1\rangle} = \sqrt{2^2+(-1)^2} = \sqrt{5}.$

(b) $\|\langle 3,4\rangle\| = \sqrt{\langle 3,4\rangle\cdot\langle 3,4\rangle} = \sqrt{3^2+4^2} = \sqrt{25} = 5$

(c) $\|\langle 1,2\rangle\| = \sqrt{\langle 1,2\rangle\cdot\langle 1,2\rangle} = \sqrt{1^2+2^2} = \sqrt{5}$

(d) $\|\langle 3,0\rangle\| = \sqrt{\langle 3,0\rangle\cdot\langle 3,0\rangle} = \sqrt{3^2+0^2} = \sqrt{9} = 3$

(ab) $\langle 2,-1\rangle\cdot\langle 3,4\rangle = 2\cdot 3+(-1)\cdot 4 = 6-4 = 2 > 0$: ACUTE ANGLE

(ac) $\langle 2,-1\rangle\cdot\langle 1,2\rangle = 2\cdot 1+(-1)\cdot 2 = 2-2 = 0$: RIGHT ANGLE

(ad) $\langle 2,-1\rangle\cdot\langle -3,0\rangle = 2\cdot(-3)+(-1)\cdot 0 = -6 < 0$: OBTUSE ANGLE

(bc) $\langle 3,4\rangle\cdot\langle 1,2\rangle = 3\cdot 1+4\cdot 2 = 3+8 = 11 > 0$: ACUTE ANGLE

(bd) $\langle 3,4\rangle\cdot\langle -3,0\rangle$: $3\cdot(-3)+4\cdot 0 = -9 < 0$: OBTUSE ANGLE

(cd) $\langle 1,2\rangle\cdot\langle -3,0\rangle$: $1\cdot(-3)+2\cdot 0 = -3 < 0$: OBTUSE ANGLE

7. (a) $(\langle 1,3\rangle \times \langle 3,1\rangle)\cdot\hat{k} = 1\cdot 1 - 3\cdot 3 = -8.$

- SINCE $-8<0$, WE TURN CLOCKWISE FROM $\langle 1,3\rangle$ TO $\langle 3,1\rangle$
- $|-8| = 8$, SO THE AREA OF THE TRIANGLE THESE VECTORS SPAN IS $\frac{1}{2}\cdot 8 = 4$.

  ↳ AREA OF PARALLELOGRAM THE VECTORS SPAN

(b) $(\langle 2,0\rangle \times \langle -1,1\rangle)\cdot\hat{k} = 2\cdot 1 - 0\cdot -1 = 2.$

- SINCE $2>0$, WE TURN COUNTERCLOCKWISE FROM $\langle 2,0\rangle$ TO $\langle -1,1\rangle$
- $|2| = 2$, SO THE AREA OF THE TRIANGLE THESE VECTORS SPAN IS $\frac{1}{2}\cdot 2 = 1$.

8. $p=(2,1) \rightsquigarrow \begin{bmatrix}2\\1\\1\end{bmatrix};\quad q=(0,3)\rightsquigarrow\begin{bmatrix}0\\3\\1\end{bmatrix};\quad r=(0,0)\rightsquigarrow\begin{bmatrix}0\\0\\1\end{bmatrix}.$

$\vec{v}=\langle -1,-1\rangle \rightsquigarrow \begin{bmatrix}-1\\-1\\0\end{bmatrix};\quad \vec{w}=\langle 4,0\rangle\rightsquigarrow\begin{bmatrix}4\\0\\0\end{bmatrix};\quad \vec{r}=\langle 0,0\rangle\rightsquigarrow\begin{bmatrix}0\\0\\0\end{bmatrix}.$

(a) $p+q+r = \begin{bmatrix}2\\1\\1\end{bmatrix}+\begin{bmatrix}0\\3\\1\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2+0+0\\1+3+0\\1+1+1\end{bmatrix} = \begin{bmatrix}2\\4\\3\end{bmatrix}$

(b) $\frac{1}{3}p+\frac{1}{3}q+\frac{1}{3}r = \frac{1}{3}\begin{bmatrix}2\\1\\1\end{bmatrix}+\frac{1}{3}\begin{bmatrix}0\\3\\1\end{bmatrix}+\frac{1}{3}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2/3+0+0\\1/3+1+0\\1/3+1/3+1/3\end{bmatrix} = \begin{bmatrix}2/3\\4/3\\1\end{bmatrix}$

(c) $p+\vec{v} = \begin{bmatrix}2\\1\\1\end{bmatrix}+\begin{bmatrix}-1\\-1\\0\end{bmatrix} = \begin{bmatrix}2-1\\1-1\\1+0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$

(d) $q+3\vec{v}-\vec{w} = \begin{bmatrix}0\\3\\1\end{bmatrix}+3\begin{bmatrix}-1\\-1\\0\end{bmatrix}-\begin{bmatrix}4\\0\\0\end{bmatrix} = \begin{bmatrix}0+3(-1)-4\\3+3(-1)-0\\1+0-0\end{bmatrix} = \begin{bmatrix}-7\\0\\1\end{bmatrix}$

(e) $\vec{r}-5\vec{v}+4\vec{w} = \begin{bmatrix}0\\0\\0\end{bmatrix}-5\begin{bmatrix}-1\\-1\\0\end{bmatrix}+4\begin{bmatrix}4\\0\\0\end{bmatrix} = \begin{bmatrix}0+5+16\\0+5+0\\0+0+0\end{bmatrix} = \begin{bmatrix}21\\5\\0\end{bmatrix}$

(f) $p+q-r = \begin{bmatrix}2\\1\\1\end{bmatrix}+\begin{bmatrix}0\\3\\1\end{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2+0-0\\1+3-0\\1+1-1\end{bmatrix} = \begin{bmatrix}2\\4\\1\end{bmatrix}$

(g) $p-\frac{1}{2}q = \begin{bmatrix}2\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0\\3\\1\end{bmatrix} = \begin{bmatrix}2-0\\1-3/2\\1-1/2\end{bmatrix} = \begin{bmatrix}2\\-1/2\\1/2\end{bmatrix}$